

# The maximum multiplicity of an eigenvalue of symmetric matrices with a given graph

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## Abstract

For a graph  $G$ ,  $M(G)$  denotes the maximum multiplicity occurring of an eigenvalue of a symmetric matrix whose zero-nonzero pattern is given by the edges of  $G$ . We introduce two combinatorial graph parameters  $T^-(G)$  and  $T^+(G)$  that give a lower and an upper bound for  $M(G)$  respectively, and we show that these bounds are sharp.

## 1 Introduction

For an  $n \times n$  symmetric matrix  $A = [a_{ij}]$ , the *graph of*  $A$ , denoted by  $G(A)$ , is the simple graph on  $n$  vertices  $1, 2, \dots, n$  where  $\{i, j\}$  is an edge of  $G(A)$  if and only if  $a_{ij} \neq 0$  for  $i \neq j$ . For a graph  $G$  on  $n$  vertices,  $S(G)$  denotes the set of all  $n \times n$  real symmetric matrices whose graph is  $G$ , and  $M(G)$  denotes the maximum multiplicity occurring of an eigenvalue of a matrix in  $S(G)$ . The *minimum rank* of  $G$ , denoted by  $\text{mr}(G)$ , is the minimum rank of  $A$  where  $A$  runs over  $S(G)$ . Note that if the multiplicity of an eigenvalue  $\lambda$

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is  $k$  for some matrix  $A$  in  $S(G)$ , then the nullity of  $A - \lambda I$  is  $k$  which implies  $\text{mr}(G) \leq \text{rank}(A - \lambda I) = n - k$ . So we can conclude that

$$M(G) = \max_{A \in S(G)} \text{nullity}(A) = n - \text{mr}(G).$$

There is a lot of interest in determining the maximum multiplicity of eigenvalues of matrices whose graph is given [7, 8, 9, 10, 11, 12, 13]. The *path cover number* of a graph  $G$ , denoted by  $P(G)$ , is the minimum number of vertex-disjoint paths needed as induced subgraphs of  $G$  that cover all the vertices of  $G$ . Duarte and Johnson in their 1999 paper [8] introduced a graph parameter  $\Delta(T)$  for a tree  $T$  to be

$$\Delta(T) := \max\{p - q \mid \text{there exist } q \text{ vertices of } T \text{ whose deletion leaves } p \text{ vertex-disjoint paths}\},$$

and showed that  $\Delta(T)$  is equal to  $M(T)$  and  $P(T)$ :

**Theorem 1.1.** [8] *For all trees  $T$ ,  $M(T) = P(T) = \Delta(T)$ .*

The definition of  $\Delta$  can be extended to any graph  $G$ . The proof of Duarte and Johnson shows that for any graph  $G$ ,  $\Delta(G)$  is a lower bound for  $P(G)$  and  $M(G)$ :

**Theorem 1.2.** [2, 8] *For all graphs  $G$ ,  $\Delta(G) \leq P(G)$  and  $\Delta(G) \leq M(G)$ .*

Later in 2004 Barioli, Fallat, and Hogben [2] pushed the results further and provided an algorithm to compute  $\Delta$ . Note from Theorem 1.2 that  $M(G)$  and  $P(G)$  are both upper bounds for  $\Delta(G)$ . But they have no relationship in general. This was observed by Barioli, Fallat, and Hogben [3, Figures 1 and 2] in the following examples: For the wheel graph  $W_5$  we have

$$P(W_5) = 2 < 3 = M(W_5)$$

and for the 5-sun  $H_5$  we have

$$M(H_5) = 2 < 3 = P(H_5).$$

From the definition of  $M(G)$ ,  $P(G)$ , and  $\Delta(G)$ , it follows that they can be computed componentwise for a disconnected graph.

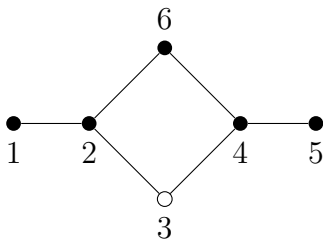


Figure 1: Graph  $G$  with  $M(G) = P(G) = \Delta(G) = 2$

**Observation 1.3.** Let  $G$  be a graph with  $k$  connected components  $G_1, G_2, \dots, G_k$ . Then  $M(G) = \sum_{i=1}^k M(G_i)$ ,  $P(G) = \sum_{i=1}^k P(G_i)$ , and  $\Delta(G) = \sum_{i=1}^k \Delta(G_i)$ .

Using the preceding observation, Theorem 1.1 can be extended to forests.

**Theorem 1.4.** If  $G$  is a forest, then  $\Delta(G) = P(G) = M(G)$ .

Note that the converse of Theorem 1.4 is not true.

**Example 1.5.** Consider the unicyclic graph  $G$  in Figure 1. We can verify that  $M(G) = P(G) = \Delta(G) = 2$ .

The preceding example shows that the equalities in Theorem 1.2 occur for some graphs with cycles in addition to trees. Indeed, for any graph  $G$  in the the following infinite family of unicyclic graphs (see Figure 2) we have  $M(G) = P(G) = \Delta(G) = 2$ .

Let  $P$  be a path on at least 5 vertices. Pick any three non-pendant consecutive vertices on  $P$ , say  $u, v$  and  $w$ . Now  $G$  is obtained from  $P$  by appending a path of length at least 2 from  $u$  to  $w$ . Clearly  $P(G) = 2$ . Deleting  $u$  and  $w$  we have  $\Delta(G) = 4 - 2 = 2$ . Furthermore, note that each  $G$  on  $n$  vertices in this family has an induced  $P_{n-1}$ . Hence  $\text{mr}(G) \geq \text{mr}(P_{n-1}) = n - 2$ . But  $\text{mr}(G) < n - 1$ , since  $G$  is not a path [5, Cor 1.5]. This shows  $\text{mr}(G) = n - 2$ , thus  $M(G) = 2$ .

In 2007 Fernandes [6] expressed  $M(G)$  for some unicyclic graphs  $G$  in terms of certain graph parameters. In 2008 AIM Minimum Rank Work Group [1] introduced the zero forcing number  $Z(G)$  for a graph  $G$  and proved that  $M(G) \leq Z(G)$  for all graphs  $G$ , where the equality holds for forests. In this article we introduce new combinatorial bounds for  $M(G)$ . Motivated by the

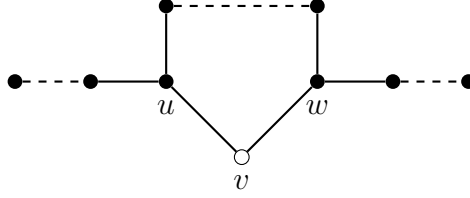


Figure 2: Graph  $G$  with  $M(G) = P(G) = \Delta(G) = 2$

definition of  $\Delta(G)$ , in Section 2 we introduce a graph parameter  $\Delta^+(G)$  in terms of path covers of  $G$  and show that

$$\Delta(G) \leq M(G) \leq \Delta^+(G),$$

for all graphs  $G$ . Then in Section 3 we introduce two more parameters  $T^-(G)$  and  $T^+(G)$  in terms of tree covers of  $G$ , and show that

$$T^-(G) \leq M(G) \leq T^+(G),$$

for all graphs  $G$  and that the bounds are sharp. In Section 4 we reduce the computation time for  $T^-$  and  $T^+$  by finding an optimal set of vertices of small size. Finally we pose some open problems in Section 5.

## 2 Graph Invariant $\Delta^+(G)$

For a graph  $G$ , we define  $\Delta^+(G)$  to be the minimum of  $p + q$  when deletion of  $q$  vertices from  $G$  leaves  $p$  vertex-disjoint paths.

**Observation 2.1.** *For any graph  $G$  on  $n$  vertices,  $\Delta^+(G) \leq n$ .*

*Proof.* Let  $S$  be an optimal set of  $q$  vertices for  $\Delta^+(G)$ . That is, deleting the  $q$  vertices in  $S$  from  $G$  leaves  $p$  disjoint paths such that  $p + q$  is minimum. Since each path has at least one vertex,  $p \leq n - q$  and then  $\Delta^+(G) = p + q \leq n$ .  $\square$

The following examples compute  $\Delta^+$  for some families of graphs.

**Example 2.2.** For the star  $S_n$  on  $n \geq 4$  vertices,  $M(S_n) = \Delta^+(S_n) = n - 2$ .

Note that  $\text{mr}(S_n) = 2$  [5, Obs 1.2] which implies  $M(S_n) = n - 2$ . Also deleting  $n - 3$  pendant vertices from a star leaves a path, viz.,  $P_3$ . Hence  $\Delta^+(S_n) = 1 + (n - 3) = n - 2$ .

**Example 2.3.** For the cycle  $C_n$  on  $n$  vertices,  $M(C_n) = \Delta^+(C_n) = 2$ .

Note that  $\text{mr}(C_n) = n - 2$  [5, Obs 1.6], hence  $M(C_n) = 2$ . Also note that to get paths induced in  $C_n$ , we need to delete at least one vertex. If deletion of  $q \geq 1$  vertices from  $C_n$  gives  $p$  paths  $P_{n_1}, \dots, P_{n_p}$ , then  $1 \leq p \leq q$ . Thus  $p + q \geq 2$ , where equality holds if and only if the number of optimal vertices deleted is 1. Thus  $\Delta^+(C_n) = 1 + 1 = 2$ .

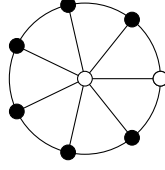


Figure 3: Wheel graph  $W_8$

**Example 2.4.** For the wheel  $W_n$  on  $n \geq 4$  vertices,  $M(W_n) = \Delta^+(W_n) = 3$ .

Note that  $\text{mr}(W_n) \geq \text{mr}(C_{n-1}) = n - 3$ , but it cannot be more than  $n - 3$  since it is neither a path nor a 2-connected linear 2-tree [5, Cor 1.5, Thm 2.26]. Hence  $\text{mr}(W_n) = n - 3$  and consequently  $M(W_n) = 3$ . For  $\Delta^+(W_n)$ , delete the vertex of degree  $n - 1$  and another vertex of degree 3 (see white vertices in Figure 3) to get  $P_{n-2}$ . So we have  $\Delta^+(W_n) = 1 + 2 = 3$ .

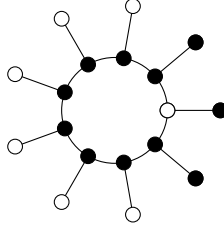


Figure 4: The 9-sun  $H_9$

**Example 2.5.** Let  $H_n$  be the  $n$ -sun. Then

$$(a) \quad M(H_n) = \begin{cases} 2 & \text{if } n = 3 \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{if } n \geq 4 \end{cases}$$

(b)  $\Delta^+(H_n) \leq n$  for  $n \geq 3$ .

Part (a) is shown in [3, Prop 3.1]. For part (b), consider  $H_3$  first. Note that deleting a vertex from the cycle leaves two paths, viz.,  $P_1$  and  $P_4$ . Thus  $\Delta^+(H_3) \leq 2 + 1 = 3$ . Now consider  $H_n$  where  $n \geq 4$ . Note that deleting a vertex from the cycle and all the pendant vertices that are at distance more than 2 from it (i.e., a total of  $n - 2$  vertices) leaves two paths, viz.,  $P_1$  and  $P_{n+1}$ . Thus  $\Delta^+(H_n) \leq 2 + (n - 2) = n$ . For an example see the 9-sun  $H_9$  in Figure 4.

Next we show that  $\Delta^+(G)$  is an upper bound for  $M(G)$  for any graph  $G$ . First we need the following lemma derived from the definition of  $M(G)$ .

**Lemma 2.6.** *Let  $G$  be a graph on  $n$  vertices. Then for all  $A \in S(G)$ ,*

$$\text{rank}(A) \geq \text{mr}(G) = n - M(G).$$

**Theorem 2.7.** *For all graphs  $G$ ,*

$$M(G) \leq \Delta^+(G).$$

*Proof.* Let  $G$  be a graph on  $n$  vertices. To show  $M(G) \leq \Delta^+(G)$ , we show that  $\text{mr}(G) = n - M(G) \geq n - \Delta^+(G)$ . Let  $A \in S(G)$ . It suffices to show that  $\text{rank}(A) \geq n - \Delta^+(G)$ . Let  $P_{n_1}, \dots, P_{n_p}$  be the vertex-disjoint paths remaining after deletion of an optimal  $q$  vertices from  $G$  such that  $\Delta^+(G) = p + q$  where  $n - q = n_1 + \dots + n_p$ . For  $i = 1, \dots, p$ , let  $B_i$  be the principle submatrix of  $A$  such that the graph of  $B_i$  is  $P_{n_i}$ . So  $B_1 \oplus \dots \oplus B_p$  is an  $(n - q) \times (n - q)$  principle submatrix of  $A$  and

$$\text{rank}(A) \geq \text{rank}(B_1 \oplus \dots \oplus B_p) = \sum_{i=1}^p \text{rank}(B_i).$$

By Theorem 1.1,  $M(P_{n_i}) = 1$  for  $i = 1, \dots, p$ . By Lemma 2.6,  $\text{rank}(B_i) \geq |P_{n_i}| - M(P_{n_i}) = n_i - 1$  for  $i = 1, \dots, p$ . Thus

$$\begin{aligned} \text{rank}(A) &\geq \sum_{i=1}^p \text{rank}(B_i) \geq \sum_{i=1}^p (n_i - 1) \\ &= \sum_{i=1}^p n_i - \sum_{i=1}^p 1 \\ &= (n - q) - p \\ &= n - (p + q) \\ &= n - \Delta^+(G). \end{aligned}$$

□

From Theorem 1.2 and 2.7, we achieve the following upper and lower bounds for  $M(G)$ :

**Corollary 2.8.**  $\Delta(G) \leq M(G) \leq \Delta^+(G)$  for all graphs  $G$ .

While Theorem 1.4 asserts that for any forest  $G$ ,  $\Delta(G) = M(G)$ , it is easy to see that even for trees  $\Delta^+$  and  $M$  do not necessarily coincide.

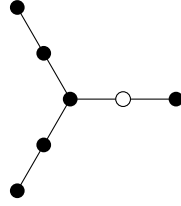


Figure 5: A generalized star

**Example 2.9.** Consider the generalized star  $G$  shown in Figure 5. Deleting the white vertex we have  $\Delta^+(G) = 3$ , but  $M(G) = P(G) = 2$ .

Examples 1.5 and 2.9 suggest that the answer to the following questions are not trivial.

**Question 2.10.** For what graphs  $G$ ,  $\Delta(G) = M(G)$ ?

The answer to this question shall contain all forests (by Theorem 1.4) and the unicyclic graphs introduced in Example 1.5.

**Question 2.11.** For what graphs  $G$ ,  $M(G) = \Delta^+(G)$ ?

The answer does not include all forests as shown in Example 2.9, but includes stars, cycles, and wheels (Example 2.2, 2.3, 2.4).

**Question 2.12.** For what graphs  $G$ ,  $\Delta(G) = M(G) = \Delta^+(G)$ ?

The answer shall include disjoint unions of stars (see Example 2.2) and paths.

### 3 Graph Invariants $T^-(G)$ and $T^+(G)$

Recall that the definitions of  $\Delta$  and  $\Delta^+$  for a graph  $G$  involve induced paths obtained by deleting vertices from  $G$ . One of the reasons for considering induced paths is that an eigenvalue of a symmetric matrix whose graph is a path has maximum multiplicity one [8]. We investigate if replacement of paths by other graphs in the definitions of  $\Delta$  and  $\Delta^+$  improves the bounds for  $M(G)$ . So we define two new graph parameters  $T^-$  and  $T^+$  as follows:

$$T^-(G) := \max \left\{ P(G \setminus S) - |S| \mid S \text{ is a subset of vertices of } G \right. \\ \left. \text{such that } G \setminus S \text{ is a forest} \right\},$$

$$T^+(G) := \min \left\{ P(G \setminus S) + |S| \mid S \text{ is a subset of vertices of } G \right. \\ \left. \text{such that } G \setminus S \text{ is a forest} \right\}.$$

Assume  $S$  is a set of  $q$  vertices, and  $G \setminus S$  is a forest which is a vertex-disjoint union of  $p$  trees  $T_1, T_2, \dots, T_p$ . Then  $P(G \setminus S) = \sum_{i=1}^p P(T_i)$  by Observation 1.3. Hence  $T^-(G)$  and  $T^+(G)$  can be rewritten as the following:

$$T^-(G) = \max \left\{ \left( \sum_{i=1}^p P(T_i) \right) - q \mid \text{there exist } q \text{ vertices of } G \text{ whose deletion} \right. \\ \left. \text{leaves } p \text{ vertex-disjoint trees } T_1, \dots, T_p \right\},$$

$$T^+(G) = \min \left\{ \left( \sum_{i=1}^p P(T_i) \right) + q \mid \text{there exist } q \text{ vertices of } G \text{ whose deletion} \right. \\ \left. \text{leaves } p \text{ vertex-disjoint trees } T_1, \dots, T_p \right\}.$$

For a forest  $G$ , the optimal set of vertices to be deleted is the empty set (i.e.,  $q = 0$ ) and consequently  $T^+(G) = T^-(G) = P(G)$ .

The following examples compute  $T^-$  and  $T^+$  for some other families of graphs.



**Example 3.1.**  $T^-(C_n) = 0$  and  $T^+(C_n) = 2$ .

Note that to get trees induced in  $C_n$ , we need to delete at least one vertex. If deletion of  $q \geq 1$  vertices from  $C_n$  gives  $p$  trees (paths)  $P_{n_1}, \dots, P_{n_p}$ , then  $1 \leq p \leq q$ . Thus  $q + \sum_{i=1}^p P(P_{n_i}) = q + p \geq 2$  and  $-q + \sum_{i=1}^p P(P_{n_i}) = -q + p \leq 0$ , where equalities hold if and only if the number of optimal vertices deleted is 1. Thus  $T^+(C_n) = 2$  and  $T^-(C_n) = 0$ .

**Example 3.2.** Let  $W_n$  be the wheel graph on  $n$  vertices. Then

- (a)  $M(W_n) = 3$ .
- (b)  $T^-(W_n) = -1, T^+(W_n) = 3$ .

The set of optimal vertices to be deleted for  $T^-$ ,  $T^+$ , and  $\Delta^+$  are shown as white vertices in Figure 3.

**Example 3.3.** Let  $H_n$  be the  $n$ -sun. Then

- (a)  $M(H_n) = \begin{cases} 2 & \text{if } n = 3 \\ \lfloor \frac{n}{2} \rfloor & \text{if } n \geq 4 \end{cases}$
- (b)  $T^-(H_n) = T^+(H_n) - 2 = \begin{cases} 1 & \text{if } n = 3 \\ \lfloor \frac{n}{2} \rfloor & \text{if } n \geq 4 \end{cases}$

An optimal set of vertices to be deleted for  $T^-$  and  $T^+$  is any single vertex from the cycle.

**Observation 3.4.** *The optimal sets for  $T^-(G)$  and  $T^+(G)$  can be chosen to be the same, when  $G$  is a wheel graph or the  $n$ -sun.*

By definitions we have the following results.

**Proposition 3.5.** *Let  $G$  be a graph. Then*

- (a)  $\Delta(G) \leq T^-(G)$  and
- (b)  $P(G) \leq T^+(G) \leq \Delta^+(G)$ .

*Proof.* (a) Suppose that deletion of  $q$  vertices from  $G$  leaves  $p$  vertex-disjoint paths  $P_{n_1}, \dots, P_{n_p}$  such that  $\Delta(G) = p - q$ . Since  $P(P_{n_i}) = 1$ ,

$$\Delta(G) = -q + \sum_{i=1}^p P(P_{n_i}) \leq T^-(G).$$

(b) Suppose that deletion of  $q$  vertices  $v_1, \dots, v_q$  from  $G$  leaves  $p$  vertex-disjoint trees  $T_1, \dots, T_p$  such that  $T^+(G) = q + \sum_{i=1}^p P(T_i)$ . Note that union of  $v_1, \dots, v_q$ , and optimal path covers of  $T_1, \dots, T_p$  forms a path cover of  $G$  of length  $T^+(G) = q + \sum_{i=1}^p P(T_i)$ . Thus  $P(G) \leq T^+(G)$ .

Let  $Q$  be an optimal set of  $q$  vertices such that deleting them from  $G$  leaves  $p$  disjoint paths  $P_1, P_2, \dots, P_p$ ; and  $\Delta^+(G) = p + q$ . Since  $P(P_i) = 1$ ,

$$\begin{aligned} \Delta^+(G) &= p + q \\ &= q + \sum_{i=1}^p 1 \\ &= q + \sum_{i=1}^p P(P_i) \\ &\geq T^+(G). \end{aligned}$$

First we note that the equality in Proposition 3.5(a) holds for forests. In fact we can show the equality for all graphs. □

**Theorem 3.6.**  $\Delta(G) = T^-(G)$  for all graphs  $G$ .

*Proof.* Let  $G$  be a graph. By Proposition 3.5 we have  $\Delta(G) \leq T^-(G)$ . So it suffices to show that  $\Delta(G) \geq T^-(G)$ . Note that  $\Delta(T) = P(T)$ , for any tree  $T$ .

Now choose an optimal set of  $q$  vertices for  $T^-(G)$  such that deleting them leaves  $p$  vertex-disjoint trees  $T_1, T_2, \dots, T_p$ . For each  $i = 1, 2, \dots, p$ , choose an optimal set of  $k_i$  vertices for  $\Delta(T_i)$  such that deleting them from  $T_i$  leaves  $\ell_i$  vertex-disjoint paths. Altogether we have chosen a set of  $(\sum_{i=1}^p k_i) + q$  vertices such that deleting them from  $G$  leaves  $\sum_{i=1}^p \ell_i$  vertex-disjoint paths. That is,

$$\begin{aligned}
\Delta(G) &\geq \left( \sum_{i=1}^p \ell_i \right) - \left( \left( \sum_{i=1}^p k_i \right) + q \right) \\
&= \left( \sum_{i=1}^p (\ell_i - k_i) \right) - q \\
&= \left( \sum_{i=1}^p \Delta(T_i) \right) - q \\
&= \left( \sum_{i=1}^p P(T_i) \right) - q \\
&= T^-(G).
\end{aligned}$$

The last equality above holds since the  $q$  vertices were chosen to be an optimal set of vertices for  $T^-(G)$ .  $\square$

Note that since  $\Delta(G) = T^-(G)$  for all graphs  $G$ , trees can be replaced by paths in the definition of  $T^-(G)$ . But this is not the case for  $T^+(G)$ . For example, for the graph  $G$  in Figure 6, we have  $T^+(G) = P(G) = 2$  and  $\Delta^+(G) = 4$ .

It is interesting to note that  $T^+(G)$  is not only just upper bound for  $P(G)$ , but also for  $M(G)$ .

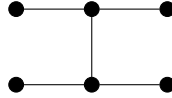


Figure 6: Graph  $G$  with  $T^+(G) = P(G) = 2$  and  $\Delta^+(G) = 4$

**Theorem 3.7.** *For any graph  $G$ ,*

$$M(G) \leq T^+(G).$$

*Proof.* The proof is similar to that of Theorem 2.7.  $\square$

Now we discuss the connection of  $T^+(G)$  with the zero forcing number  $Z(G)$  which is an well-known upper bound for  $M(G)$ . A zero forcing set  $Z$  is

a subset of vertices of  $G$  such that if the vertices in  $Z$  are initially colored and the other vertices are not colored, then all the vertices of  $G$  become colored after we apply the following coloring rule: if  $u$  is a colored vertex with exactly one uncolored neighbor  $v$ , then color  $v$ . The zero forcing number  $Z(G)$  is the minimum size of a zero forcing set of  $G$ . For example, in the graph  $G$  in Figure 7, two pendant vertices at distance 3 form a zero forcing set because if they are initially colored, then they will force the remaining vertices to be colored. Thus  $Z(G) \leq 2$ . Since there is no zero forcing set of size 1, we have  $Z(G) = 2$ .

Let  $S$  be an optimal set for  $T^+(G)$ . Then  $G \setminus S$  is a forest for which  $P(G \setminus S) = M(G \setminus S) = Z(G \setminus S)$ . Now we can find a zero forcing set  $Z'$  of  $G \setminus S$  of size  $P(G \setminus S)$  by choosing an endpoint of each path in a minimum path cover of  $G \setminus S$ . It can be verified that  $Z' \cup S$  is a zero forcing set of  $G$  and consequently

$$Z(G) \leq |Z' \cup S| = |Z'| + |S| = P(G \setminus S) + |S| = T^+(G).$$

Although  $T^+(G)$  does not improve the upper bound  $Z(G)$  of  $M(G)$ , it has a different approach than the zero forcing number  $Z(G)$ .

**Observation 3.8.** *For any graph  $G$ ,  $Z(G) \leq T^+(G)$ .*

Note that the equality does not hold for the graph in Figure 7, where  $Z(G) = P(G) = 2$  and  $T^+(G) = 4$ .

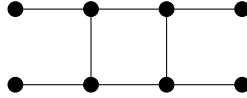


Figure 7: Graph  $G$  with  $Z(G) < T^+(G)$ .

Now we summarize earlier results involving different parameters in the following corollary.

**Corollary 3.9.** *For all graphs  $G$ ,*

$$\Delta(G) = T^-(G) \leq M(G) \leq Z(G) \leq T^+(G) \leq \Delta^+(G),$$

*where  $T^-(G) = M(G) = T^+(G)$  if  $G$  is a forest.*

*Proof.* By Theorem 3.6, we have  $\Delta(G) = T^-(G)$ . By Theorem 1.2, we have  $\Delta(G) \leq M(G)$ , hence  $T^-(G) \leq M(G)$ . By Theorem 3.7 and Observation 3.8,  $M(G) \leq Z(G) \leq T^+(G)$ . Finally, by Proposition 3.5(b),  $T^+(G) \leq \Delta^+(G)$ .

Note that if  $G$  is a forest, then an optimal set of vertices to be deleted for  $T^-$  and for  $T^+$  can be chosen to be the empty set. Hence  $q = 0$  and  $\sum_{i=1}^p P(T_i) = P(G)$ . Hence,  $T^-(G) = T^+(G) = P(G) = M(G)$ .  $\square$

Note that since  $T^-(H_5) = M(H_5)$  and  $M(W_5) = T^+(W_5)$ ,  $T^-(G)$  and  $T^+(G)$  give a tight lower bound and a tight upper bound for  $M(G)$  respectively.

## 4 On computing $T^-$ and $T^+$

In this section we provide some tools to reduce the time of computation for  $T^-$  and  $T^+$  for graphs. Let  $G$  be a graph on  $n$  vertices with  $m$  edges. We show that there is always an optimal set of vertices to be deleted for  $T^-$  and  $T^+$  that has size at most  $m - n + 1$ .

An *Eulerian subgraph* of  $G$  is a subgraph of  $G$  whose vertices have even degree. It is well-known that an Eulerian subgraph  $H$  is a union of cycles in  $H$ . The (binary) *cycle space* of  $G$  is the set of Eulerian subgraphs in  $G$ . The cycle space of  $G$  can be described as a vector space over  $\mathbb{Z}_2$ . A basis for this vector space is called a *cycle basis* of the graph. It can be shown that the dimension of the cycle space of a connected graph is  $m - n + 1$  [4, Section 1.9 pp. 23–28]. Therefore any cycle in  $G$  is a linear combination of cycles in a cycle basis and each of  $m - n + 1$  cycles in a cycle basis is not a linear combination of smaller cycles.

**Lemma 4.1.** *Let  $G$  be a connected graph on  $n$  vertices with  $m$  edges. Let  $S$  be a set of vertices of  $G$  such that  $G \setminus S$  does not have any cycles. Then there is a set  $S' \subseteq S$  with  $|S'| \leq m - n + 1$  such that  $G \setminus S'$  does not have any cycles.*

*Proof.* If  $|S| \leq m - n + 1$ , then choose  $S' = S$ . Otherwise, fix a cycle basis of  $G$ :

$$B = \{C_1, C_2, \dots, C_{m-n+1}\}.$$

Let  $S_1$  be the subset of  $S$  consisting of vertices  $v$  that is on exactly one cycle in  $B$ . Note that  $S_1$  might be empty. Let  $B_1$  be the subset of  $B$  corresponding to vertices of  $S_1$ . Note that  $|S_1| = |B_1|$  and a cycle basis of

$G \setminus S_1$  is  $B \setminus B_1$ . Let  $S_2$  be the subset of  $S \setminus S_1$  consisting of at most one vertex from each cycle in  $B \setminus B_1$  such that  $(G \setminus S_1) \setminus S_2$  does not have any cycle. Choose  $S' = S_1 \cup S_2$ . Then  $G \setminus S'$  does not have any cycles and  $|S'| = |S_1| + |S_2| \leq |B_1| + |B \setminus B_1| = |B| = m - n + 1$ .  $\square$

**Lemma 4.2.** *Let  $G$  be a graph on  $n$  vertices with  $m$  edges. Let  $S$  be an optimal set of vertices for  $T^+(G)$  (respectively  $T^-(G)$ ). Then any set  $S' \subseteq S$  such that  $G \setminus S'$  does not have a cycle is also an optimal set of vertices for  $T^+(G)$  (respectively  $T^-(G)$ ).*

*Proof.* First note, by the definitions of  $T^+(G)$  (respectively  $T^-(G)$ ), that  $G \setminus S$  is a forest and  $T^+(G) = T^+(G \setminus S) + |S|$  (respectively  $T^-(G) = T^-(G \setminus S) - |S|$ ).

Consider  $S' \subseteq S$  such that  $G \setminus S'$  is a forest  $F$ . Let  $R = S \setminus S'$ . Choose an optimal path cover of  $F \setminus R = G \setminus S$ :

$$P = \{P_1, P_2, \dots, P_k\}.$$

Then  $P$  together with vertices in  $R$  forms a path cover for  $F$  which implies

$$T^+(F \setminus R) + |R| \geq T^+(F).$$

If  $T^+(F \setminus R) + |R| > T^+(F)$ , then

$$T^+(G) = T^+(G \setminus S) + |S| = T^+(F \setminus R) + |R| + |S'| > T^+(F) + |S'|.$$

This contradicts the optimality of  $S$ . Thus,

$$T^+(F \setminus R) + |R| = T^+(F).$$

Consequently,

$$T^+(G) = T^+(F \setminus R) + |R| + |S'| = T^+(F) + |S'|.$$

That is,  $S'$  is also an optimal set of vertices for  $T^+(G)$ .

Similarly, for  $T^-(G)$  consider  $S' \subseteq S$  such that  $G \setminus S'$  is a forest  $F$ . Let  $R = S \setminus S'$ . Choose an optimal path cover of  $F \setminus R = G \setminus S$ :

$$P = \{P_1, P_2, \dots, P_k\}.$$

Then  $P$  together with vertices in  $R$  forms a path cover for  $F$  which implies

$$T^-(F \setminus R) - |R| \leq T^-(F).$$

If  $T^-(F \setminus R) - |R| < T^-(F)$ , then

$$T^-(G) = T^-(G \setminus S) - |S| = T^-(F \setminus R) - |R| - |S'| < T^-(F) - |S'|.$$

This contradicts the optimality of  $S$ . Thus,

$$T^-(F \setminus R) - |R| = T^-(F).$$

Consequently,

$$T^-(G) = T^-(F \setminus R) - |R| - |S'| = T^-(F) - |S'|.$$

That is,  $S'$  is also an optimal set of vertices for  $T^-(G)$ .  $\square$

**Proposition 4.3.** *Let  $G$  be a connected graph on  $n$  vertices with  $m$  edges. The optimal sets of vertices for  $T^+$  and  $T^-$  can be chosen so that each of them has at most  $m - n + 1$  vertices.*

*Proof.* Consider an optimal set  $S$  of vertices for  $T^+(G)$  (respectively  $T^-(G)$ ). Then  $G \setminus S$  is a forest. By Lemma 4.1, there is  $S' \subseteq S$  such that  $|S'| \leq m - n + 1$  and  $G \setminus S'$  does not have any cycles. Then by Lemma 4.2,  $S'$  is also an optimal set of vertices for  $T^+(G)$  (respectively  $T^-(G)$ ).  $\square$

## 5 Open Problems

Recall from Corollary 3.9 and Proposition 3.5(b) that  $T^-(G)$  and  $T^+(G)$  are lower and upper bounds for both  $M(G)$  and  $P(G)$  respectively. Therefore if  $T^-(G) = T^+(G)$ , then

$$T^-(G) = \Delta(G) = P(G) = M(G) = Z(G) = T^+(G).$$

So it is natural to seek characterization of graphs  $G$  for which  $T^-(G) = T^+(G)$ .

**Question 5.1.** *For what graphs  $G$ ,  $T^-(G) = T^+(G)$ ?*

Note that  $T^-(G) = T^+(G)$  for all forests (see Corollary 3.9) and all unicyclic graphs in Example 1.5.

By Theorem 3.7,  $M(G) \leq T^+(G)$  for all graphs  $G$  and the equality holds for forests (see Corollary 3.9), cycles (see Examples 2.3, 3.1), wheel graphs (see Example 3.2), and all unicyclic graphs in Example 1.5. It may be interesting to know what other graphs give the equality.

**Question 5.2.** *Characterize graphs  $G$  for which  $M(G) = T^+(G)$ .*

Similarly by Proposition 3.5(b),  $P(G) \leq T^+(G)$  for all graphs  $G$  and the equality holds for forests (see Corollary 3.9) and all unicyclic graphs in Example 1.5. It may be worth exploring the graphs for which the equality holds.

**Question 5.3.** *Characterize graphs  $G$  for which  $P(G) = T^+(G)$ .*

Finally, note that for any graph  $G$  we have  $Z(G) \leq T^+(G)$  (Observation 3.8), while the equality holds for forests (see Corollary 3.9), cycles (see Examples 2.3, 3.1), wheel graphs (see Example 3.2), and all unicyclic graphs in Example 1.5. But the equality does not always hold (see Figure 7). It might be of interest to classify graphs for which we have the equality.

**Question 5.4.** *Characterize graphs  $G$  for which  $Z(G) = T^+(G)$ .*

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## References

- [1] AIM Minimum Rank Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelsen, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428:1628–1648, 2008.
- [2] F. Barioli, S. Fallat, and L. Hogben. Computation of minimal rank and path cover number for certain graphs. *Linear Algebra and its Applications*, 392:289–303, 2004.
- [3] F. Barioli, S. Fallat, and L. Hogben. On the difference between the maximum multiplicity and path cover number for tree-like graphs. *Linear Algebra and its Applications*, 409:13–31, 2005.
- [4] R. Diestel. *Graph Theory*. Graduate Texts in Mathematics 173. Springer, 2012.



- [5] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: A survey. *Linear Algebra and its Applications*, 426:558–582, 2007.
- [6] R. Fernandes. On the maximum multiplicity of an eigenvalue in a matrix whose graph contains exactly one cycle. *Linear Algebra and its Applications*, 422:1–16, 2007.
- [7] J. Genin and J.S. Maybee. Mechanical vibration trees. *Journal of Mathematical Analysis and Applications*, 45:476–763, 1974.
- [8] C.R. Johnson and A. Leal-Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. *Linear and Multilinear Algebra*, 46:139–144, 1999.
- [9] C.R. Johnson and A. Leal-Duarte. On the possible multiplicities of the eigenvalues of an hermitian matrix whose graph is a given tree. *Linear Algebra and its Applications*, 348:7–21, 2002.
- [10] C.R. Johnson and C.M. Saiago. Estimation of the maximum multiplicity of an eigenvalue in terms of the vertex degrees of the graph of a matrix. *Electronic Journal of Linear Algebra*, 9:27–31, 2002.
- [11] P.M. Nylen. Minimum-rank matrices with prescribed graph. *Linear Algebra and its Applications*, 248:303–316, 1996.
- [12] S. Parter. On the eigenvalues and eigenvectors of a class of matrices. *Journal of the Society for Industrial and Applied Mathematics*, 8:376–388, 1960.
- [13] G. Wiener. Spectral multiplicity and splitting results for a class of qualitative matrices. *Linear Algebra and its Applications*, 1984:15–29, 61.